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Numerical Study of the Stochasticity of Dynamical Systems with More Than Two Degrees of Freedom

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Dynamical Systems with N degrees of freedom can be reduced, by the method of surfaces of section, to the study of a $(2N-2)$ dimensional mapping. We consider here, as a model problem, the mapping given, for $N=3$, by the following equations:

$$T \begin{cases} x_1 = x_0 + a_1 \sin(x_0 + y_0) + b \sin(x_0 + y_0 + z_0 + t_0) \\ y_1 = x_0 + y_0 \\ z_1 = z_0 + a_2 \sin(z_0 + t_0) + b \sin(x_0 + y_0 + z_0 + t_0) \\ t_1 = z_0 + t_0 \end{cases} \pmod{2\pi}$$

The purpose of the present paper is to test numerical methods for the study of the stochasticity of this dynamical system. Therefore, we study the two largest eigenvalues—in absolute magnitude—of the tangential linear mapping of T^n. We then meet important precision problems on the computer, and use both a standard method with multiprecision computations and an entirely new one. The method enables us to show numerically that the mapping T is indeed close to a C-system, in the ergodic zone. This method can be generalized easily to systems with a small number N of degrees of freedom, and becomes then a numerical tool for the study of the stochasticity of such dynamical systems.

1. INTRODUCTION

Many studies have been made these last few years on the motion of a star in an axisymmetric galaxy or in the plane of symmetry of a spiral galaxy [4, 10]. This is an instance of a dynamical system with two degrees of freedom. The present study can be related to the study of a motion of a star in a galaxy without any symmetry. In fact, we study dynamical systems with three degrees of freedom, and more especially the ergodic properties of such systems. It will also be shown that an extension of this study to systems with a larger number of degrees of freedom ($N = 4, 5, 6...$) is possible. These systems are not purely ergodic, but are close to integrable systems. The work of Arnold [3] and Moser [9] has shown that they are, if certain conditions are satisfied, behaving like the integrable

case in certain regions of the phase space. However, the stochastic properties, in the nonintegrable case, are unknown.

One of the most fruitful methods for studying numerically such systems is the use of "surfaces of section." If we consider, quite generally, a conservative system with N degrees of freedom, the corresponding phase space has $2N$ dimensions and a given trajectory must lie on a manifold with $2N - 1$ dimensions, corresponding to a given value of energy. In this manifold, we define a "surface of section" as a given manifold with $M = 2N - 2$ dimensions, and we consider the successive intersections of the trajectory with this surface of section. It is useful to define a mapping T of the surface of section on itself and to study such a mapping as a model problem, as it lends itself very well to numerical experiments.

In a previous paper [6], Froeschle studied some ergodic properties of a two-dimensional mapping T, and the variation with the number n of iterations, of the largest eigenvalue λ_1^n (in absolute magnitude) of the tangential linear mapping T^{n*}. He found that the numerical results seem to follow the general behavior of the C-systems, which are dynamical systems with special stochastic properties.

We reproduce here the definition of the C-systems given by Arnold and Avez [1]. Let ϕ be a diffeomorphism of the class C^2 of a compact connex Riemannian manifold M of the class C^∞. Let φ^* be the linear tangential mapping of ϕ from TM_m into $TM_{\phi(m)}$ where TM_m and $TM_{\phi(m)}$ are the tangential spaces to m and $\phi(m)$. Then we say that ϕ is a C-diffeomorphism if:

(1) TM_m is the direct sum of two tangential spaces X_m and Y_m of positive dimension, i.e., $TM_m = X_m \oplus Y_m$.

(2) for all i

$$\|(\phi^i)^* \xi\| \geqslant a\, e^{\lambda i} \|\xi\|, \qquad \|(\phi^{-i})^* \xi\| \leqslant b\, e^{-\lambda i} \|\xi\| \quad \text{if} \quad \xi \in X_m,$$

and

$$\|(\phi^i)^* \xi\| \leqslant b\, e^{-\lambda i} \|\xi\|, \qquad \|(\phi^{-i})^* \xi\| \geqslant a\, e^{\lambda i} \|\xi\| \quad \text{if} \quad \xi \in Y_m,$$

where the positive constants a, b, λ do not depend on i and ξ. By $\|\xi\|$ we mean the length of the vector ξ. For example, on the torus $M = \{x, y\}$ (mod 1) with the Riemannian metric $ds^2 = dx^2 + dy^2$ we consider the mapping:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (\text{mod } 1).$$

Then the above definition holds with:

$$a = b = 1,$$
$$e^{\lambda} = \lambda_1,$$
$$e^{-\lambda} = \lambda_2,$$

λ_1 and λ_2 being the eigenvalues such that $0 < \lambda_2 < 1 < \lambda_1$. Therefore this mapping is a C-diffeomorphism.

The same quantity $\lambda_1{}^n$ has been studied, but for a four dimensional mapping which has been taken as a model problem for the study of a dynamical system with three degrees of freedom [7]. For such systems, it is also necessary to study the variation of the second eigenvalue $\lambda_2{}^n$, in decreasing absolute magnitude, of the tangential linear mapping. Froeschle studied only the behavior of the first eigenvalue, as he met important numerical problems, due in particular to the limited number of significant digits on the computer, and due also to the method used.

Our purpose, in this paper, is to present two methods for this study. These methods have been tested on the Nice Observatory IBM 7040 computer (basic cycle = 8 μ sec; 32 K words of memory; 8 significant digits in floating point simple precision). In Section II, we give the equations of the mapping, we recall some relevant results and the problem is studied with a multiprecision program. In Section III, we give a new approach of the problem by a special numerical method, which is valid in general for systems with N degrees of freedom. In Section IV, the results are given.

II. Study of the Problem in Simple and Multiprecision

A previous paper [6] has shown for a two dimensional mapping T_1 that if $\lambda_1{}^n$ is the largest eigenvalue of the tangential linear mapping T_1^{n*} of $T_1{}^n$, then the quantities

$$\theta_1{}^n = (\log_{10} | \lambda_1{}^n |)/n$$

and the Cesaro mean $\mu_1{}^n$ of $\theta_1{}^n$ are good estimators of stochasticity. In this paper we use the same quantity but for a four-dimensional mapping T, ($M = 4$).

The mapping T of the (x, y, z, t) space over itself is defined by:

$$T \begin{cases} x_1 = x_0 + a_1 \sin(x_0 + y_0) + b \sin(x_0 + y_0 + z_0 + t_0) \\ y_1 = x_0 + y_0 \\ z_1 = z_0 + a_2 \sin(z_0 + t_0) + b \sin(x_0 + y_0 + z_0 + t_0) \\ t_1 = z_0 + t_0 \end{cases} \pmod{2\pi} \qquad (1)$$

The determinant of the Jacobian matrix is equal to 1. This mapping has been suggested by Arnold [2]. If $b = 0$, then the mapping T is the product of two area-preserving mappings T_1 of (x, y) on itself and T_2 of (z, t) on itself. These mappings are the inverse mappings of those given by Taylor [11]. Each of these mappings displays the well-known features of systems with two degrees of freedom

[8]. Since the set of points obtained by repeated applications of the mapping lies on the torus:

$$\tau_M = \{(x, y, z, t) \pmod{2\pi}\},$$

escape cannot occur, and so, limit numerical investigations.

Our desire is to establish here a property of this mapping, which will be useful later. Equations (1) give the inverse mapping T^{-1} as

$$T^{-1}\begin{cases} x_1 = x_0 - a_1 \sin(y_0) - b \sin(y_0 + t_0) \\ y_1 = -x_0 + y_0 + a_1 \sin(y_0) + b \sin(y_0 + t_0) \\ z_1 = z_0 - a_2 \sin(t_0) - b \sin(y_0 + t_0) \\ t_1 = -t_0 + z_0 + a_2 \sin(t_0) + b \sin(y_0 + t_0) \end{cases} \pmod{2\pi} \tag{2}$$

Let us consider the transformation S given by the equations

$$S\begin{cases} x' = x \\ y' = -(x + y) \\ z' = z \\ t' = -(z + t) \end{cases} \tag{3}$$

Then we can verify that

$$S \circ T = T^{-1} \circ S,$$

hence

$$T^{-1} = S \circ T \circ S^{-1}$$

and

$$T^{-n} = S \circ T^n \circ S^{-1} \tag{4}$$

This relation proves that the characteristic equation of the Jacobian matrix L_n of T^n is reciprocal, by the following argument: Let T^{n*} be the linear tangential mapping of T^n. We have the relation

$$T^{(-n)*} = S \circ T^{n*} \circ S^{-1} \tag{5}$$

Then according to a well-known result, $T^{(-n)*}$ and T^{n*} have the same eigenvalues. So that if $\lambda_i{}^n$ is an eigenvalue of T^{n*}, then $1/\lambda_i{}^n$ is also an eigenvalue and the characteristic equation of the Jacobian matrix $L_n(P_0)$ which represents the linear tangential mapping T^{n*} of T^n at P_0 is *reciprocal* (P_0 being the initial point). It will therefore be sufficient to study the behavior of just the two largest eigenvalues (that is largest in absolute magnitude) of the tangential linear mapping T^{n*}.

We must first note that the *composite mapping theorem* has been used to compute

the Jacobian matrix $L_n(P_0)$, which represents T^{n*} at the initial point P_0, i.e.,

$$T^{n*}(P_0) = T^*(T^{n-1}(P_0)) \circ T^{(n-1)*}(P_0) \tag{6}$$

Since the elements of the matrix

$$L_n(P_0) = D = (d_{ij}\,; \quad i = 1, M; \quad j = 1, M),$$

in the studied case (ergodic case) exceed rapidly the largest floating point value on the computer ($\sim 10^{38}$ on IBM 7040), we have used the following device as often as necessary: if the largest eigenvalue $|\lambda_1^n|$ becomes larger than 10^2, we divide each term of the matrix by 10^2, and take:

$$|\lambda_1^n| = 10^{2m} |\lambda_1'^n|$$

where m is the number of times the terms have been divided by this quantity, and $|\lambda_1'^n|$ the largest eigenvalue of the new matrix.
By using this device, we introduce in fact a "*floating-point*" representation of the numbers, which is different than that used by the computer, and which enables us to handle larger quantities.
To compute $|\lambda_1^n|$ for the n-th iteration, we solve the characteristic equation (which is reciprocal):

$$\lambda^4 + C_1\lambda^3 + C_2\lambda^2 + C_1\lambda + 1 = 0$$

We find numerically that the coefficients C_1 and C_2 are such that

$$C_1^2 \gg |C_2| \gg |C_1| \gg 1$$

We obtain the approximate solutions:

$$|\lambda_1^n| \simeq |C_1|$$
$$|\lambda_2^n| \simeq |C_2| / |C_1|$$

corresponding to the two largest eigenvalues.
The two others are:

$$|\lambda_3^n| = 1/|\lambda_2^n|$$
$$|\lambda_4^n| = 1/|\lambda_1^n|$$

because the characteristic equation is reciprocal.

However, if we study the asymptotic behavior of the two largest eigenvalues, we

meet the following problem: $L_n(P_0)$ being the matrix under study, the coefficients C_1 and C_2 are

$$C_1 = -\sum_{i=1}^{M} d_{ii},$$

$$C_2 = \sum_{i=1}^{M} \sum_{k=1}^{M-i} (d_{ii} \times d_{i+k,i+k} - d_{i,i+k} \times d_{i+k,i}).$$

We have

$$d_{ii} \approx C_1 \qquad \text{(of the order of } C_1\text{)},$$
$$d_{ii} \times d_{i+k,i+k} \approx C_1^2.$$

If ϵ is the relative error on the computer, the absolute error on C_2 is $\epsilon\, C_1^2$. The absolute error on $|\lambda_2^n| \simeq |C_2|/|C_1|$ is then $\epsilon |C_1| \approx \epsilon |\lambda_1^n|$. The relative error on $|\lambda_2^n|$ is $\epsilon |\lambda_1^n|/|\lambda^n{}_2|$.
The results obtained for $|\lambda_2^n|$ will be rather bad if $|\lambda_1^n| \gg |\lambda_2^n|$. They will be obviously nonsignificant if the following relation holds:

$$|\lambda_1^n|/|\lambda_2^n| \geqslant 1/\epsilon$$

That is to say if we have in simple precision on the IBM 7040

$$|\lambda_1^n|/|\lambda_2^n| \geqslant 10^8.$$

We set

$$\psi_i^n = \log_{10} |\lambda_i^n| \qquad i = 1, 2$$

Figure 1a shows ψ_1^n and ψ_2^n versus n, for $n = 1(1)1000$. We can observe that the two curves first go apart from each other for n increasing till 300. Then, for $n > 300$, $|\lambda_2^n|$ becomes completely erroneous and reflects only the round-off errors due to $|\lambda_1^n|$. We obtain

$$6.0 \leqslant \psi_1^n - \psi_2^n \leqslant 11.4$$

continuously. (The relative error on the computer is $\epsilon_1 \simeq 10^{-8}$.) We tried also in simple precision the "accelerated Rutishauser method," for the computation of these eigenvalues, as described in [5] where the same phenomenon is observed.

After translating the program to double precision, which gives us 16 significant digits, it appears that

$$14.0 \leqslant \psi_1^n - \psi_2^n \leqslant 19.4 \qquad \text{for} \quad n > 600$$

continuously.

Figure 1b shows a divergence of the two curves ψ_1^n and ψ_2^n for $n \leqslant 600$ and then a "parallelism" when $n > 600$, which depends on the new relative error of the computer ($\epsilon_2 \simeq 10^{-16}$). We observe that the interval on which the curves diverge is, when the precision is doubled, twice as large.

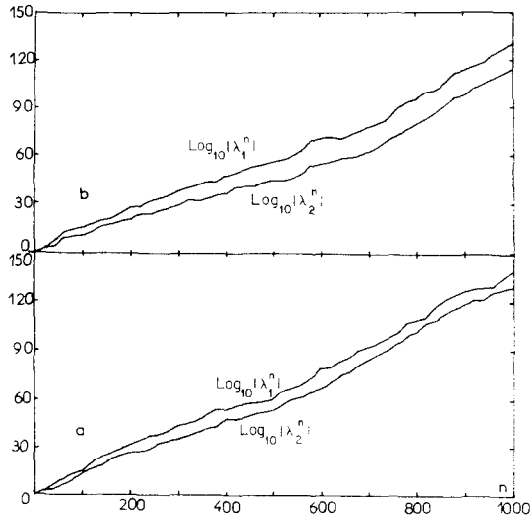

FIG. 1. Eigenvalues of the four-dimensional mapping (1). Upper curve: variation of $\log_{10} | \lambda_1^n |$ with n. Lower curve: variation of $\log_{10} | \lambda_2^n |$ with n. The initial conditions are: $x_0 = 2.0$, $y_0 = 0.$, $z_0 = 2.1$, $t_0 = 0$. The parameters of the mapping are: $a_1 = -1.3$, $a_2 = -1.3$, $b = 0.01$. This corresponds to an ergodic zone. (a) Computation in simple precision (8 significant digits). (b) Computation in double precision (16 significant digits).

What we want to see now is, if by using a greater precision, we continue to observe the phenomenon. Hardware extended precision not being available on the computer, we used the SHARE library multiprecision subroutines (SDA N° 3477–3482).

These programs have been coded in FORTRAN IV, and are independent of the computer used. The characteristics of the machine (such as the number of bits per word, or the number of significant digits for a floating point variable or constant) are input parameters. The number MP of words (MP $\geqslant 3$) used for a variable in multiprecision is also an input parameter. When working with *MP* words, the variables are represented in normalized form by an integer array of MP words. In the first word, we find the exponent with its sign. In the other words, we find the significant digits. Each of these words contains the variable sign. The exponent can take values from $-(2^{35} - 1)$ to $(2^{35} - 1)$.

We used 6 basic subroutines written in FORTRAN IV:

ENJKPM converts a variable from single precision to multiprecision;

EXJKPM converts a variable from multiprecision to single precision;

SVJKPM adds two variables in multiprecision and puts their sum in a variable in multiprecision;

SUJKPM subtracts two variables in multiprecision and puts their difference in a variable in multiprecision;

MUJKPM multiplies two variables in multiprecision and puts their product in a variable in multiprecision;

DIJKPM divides two variables in multiprecision and puts their quotient in a variable in multiprecision.

Other subroutines exist for the computation of usual mathematical functions such as COS, SIN, EXP, LOG, SQRT, etc.

The most important parts of our program have been rewritten in multiprecision with

$$\text{MP} = 3, \quad \text{i.e., } 20 \text{ significant digits, and also}$$
$$\text{MP} = 4, \quad \text{i.e., } 30 \text{ significant digits.}$$

The time necessary for processing the new program with MP = 4 is rather long (about ten times the time necessary for double precision, that is to say one hour for 1000 iterations on IBM 7040), because the main loop of the program has then a very important number of FORTRAN instructions. We obtain for MP = 3

$$18.0 \leqslant \psi_1{}^n - \psi_2{}^n \leqslant 23.4,$$

and for MP = 4

$$28.0 \leqslant \psi_1{}^n - \psi_2{}^n \leqslant 33.4.$$

Figures 2a and 2b show indeed a divergence of the two curves as soon as $n = 1$, and then again a "parallelism" which occurs for increasing values of n as the precision becomes greater, as expected. We must however note that the shape of the curves in multiprecision is not exactly the same as that observed in simple, or double precision. This is not very significant. Since the round-off procedures are not the same, slight differences are introduced in the coordinates of the transformed points; and it is characteristic of dynamical systems near of C-systems that two orbits, initially close to each other, diverge exponentially. It is highly probable that the two largest eigenvalues $|\lambda_1{}^n|$ and $|\lambda_2{}^n|$ of T^{n*} will continue to diverge

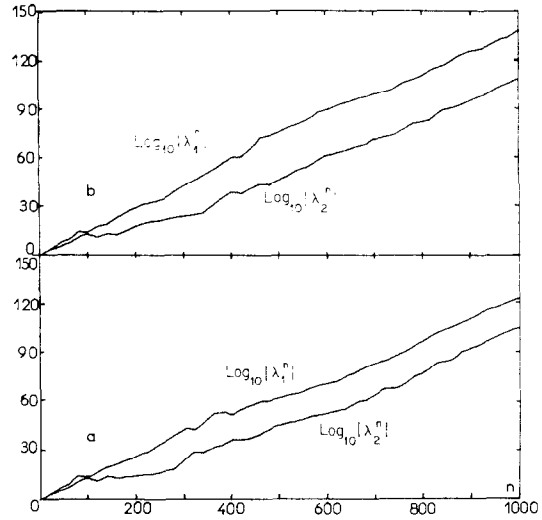

FIG. 2. Same conditions as Fig. 1. (a) Computation in triple precision (20 significant digits). (b) Computation in quadruple precision (30 significant digits).

exponentially for large n, but this cannot be shown with the present method, because of the limited precision used.

These results suggest that the method we have used so far is not suited to the problem.

III. A New Approach of the Problem

Let L_n be the $M \times M$ matrix of the linear tangential mapping T^{n*}. Let us call $(\lambda_j{}^n; j = 1, M)$ its eigenvalues, $\lambda_1{}^n$ being the largest in absolute value, and $(\mathbf{V}_j{}^n ; j = 1, M)$, the corresponding eigenvectors. We have

$$L_n \mathbf{V}_j{}^n = \lambda_j{}^n \mathbf{V}_j{}^n. \tag{7}$$

We also suppose that

$$|\lambda_j{}^n| / |\lambda_{j+1}^n| > 10^8 \qquad j = 1, \quad M - 1 \tag{8}$$

and that we have computed directly $\lambda_j{}^n$ and $\mathbf{V}_j{}^n$. Our purpose is to compute with the precision of at least $\epsilon_1 \simeq 10^{-8}$ the λ_j^{n+1} and the $\mathbf{V}_j^{n+1}$. As the eigenvectors $\mathbf{V}_j{}^n$ are independent, we can write

$$\mathbf{V}_j^{n+1} = \sum_{k=1}^{M} a_{jk} \mathbf{V}_k{}^n. \tag{9}$$

We take Z as the matrix of the linear tangential mapping T^* at the point $P_n = T^n(P_0)$, and introduce the vectors W_j^{n+1} by

$$\mathbf{W}_j^{n+1} = Z\mathbf{V}_j{}^n. \tag{10}$$

The matrix Z is known, but the matrix

$$A = (a_{jk}\,, j = 1, M; k = 1, M)$$

is unknown.
We have

$$L_n\mathbf{V}_j^{n+1} - \sum_{k=1}^{M} a_{jk}\lambda_k{}^n\mathbf{V}_k{}^n \tag{11}$$

Using the "*composite mapping theorem,*" we derive

$$L_{n+1}\mathbf{V}_j^{n+1} = ZL_n\mathbf{V}_j^{n+1} \tag{12}$$

and

$$L_{n+1}\mathbf{V}_j^{n+1} = Z\left(\sum_{k=1}^{M} a_{jk}\lambda_k{}^n\mathbf{V}_k{}^n\right), \tag{13}$$

$$L_{n+1}\mathbf{V}_j^{n+1} = \sum_{k=1}^{M} a_{jk}\lambda_k{}^n\mathbf{W}_k^{n+1}. \tag{14}$$

But we also have

$$L_{n+1}\mathbf{V}_j^{n+1} = \lambda_j^{n+1}\mathbf{V}_j^{n+1}. \tag{15}$$

If we set

$$\mathbf{V}_j^{n+1} = \sum_{k=1}^{M} b_{jk}\mathbf{W}_k^{n+1}, \tag{16}$$

we have

$$\lambda_j^{n+1}\sum_{k=1}^{M} b_{jk}\mathbf{W}_k^{n+1} = \sum_{k=1}^{M} a_{jk}\lambda_k{}^n\mathbf{W}_k^{n+1}. \tag{17}$$

Since the $\mathbf{W}_k^{n+1}$ are independant vectors, we have the fundamental relationships

$$b_{jk} = (\lambda_k{}^n/\lambda_j^{n+1})\,a_{jk}\,. \tag{18}$$

In order to use the Eq. (18), we estimate the order of magnitude of the a_{jk} and

b_{jk} comparatively to $1/\epsilon_1$. The norms of the eigenvectors ($\mathbf{V}_j^n$, $\mathbf{V}_j^{n+1}$; $j = 1, M$) are equal to one, hence from Eq. (9) the a_{ij} are small compared to $1/\epsilon_1$. From Eq. (10), the norms of the vectors W_j^{n+1} are such that

$$\frac{1}{|\operatorname{Tr} Z|} \leqslant \| W_j^{n+1} \| \leqslant |\operatorname{Tr} Z|,$$

Tr(Z) being of the order of magnitude of the largest eigenvalue and 1/TrZ of the order of magnitude of the smallest one. Therefore, from the Eq. (16) max $|b_{jk}|$ is of the order of $|\operatorname{Tr} Z|$ which is still small compared to $1/\epsilon_1$.

Using the fact that the eigenvalues of L_n differ by a very large amount when we start the method, i.e.,

$$\frac{|\lambda_2^n|}{|\lambda_1^n|} < \epsilon_1$$

and since

$$|\lambda_1^{n+1}| > |\lambda_1^n|$$

from the direct calculation, we also have

$$|\lambda_2^n| / |\lambda_1^{n+1}| < \epsilon_1 .$$

On the other hand, we have (see Section II)

$$\lambda_4^n = 1/\lambda_1^n \qquad \text{and} \qquad \lambda_3^n = 1/\lambda_2^n.$$

Therefore, for $k > j$ we have

$$|\lambda_k^n| / |\lambda_j^{n+1}| < \epsilon_1 ,$$

and from Eqs. (18) we have the b_{jk} of the order of ϵ_1, since the a_{jk} and the b_{jk} are small compared to $1/\epsilon_1$. These b_{jk} will be taken equal to zero in the following algebra. By the same arguments, we take for

$$k < j \qquad a_{jk} \simeq 0$$

Thus we obtain the fundamental relationship

$$\mathbf{V}_j^{n+1} = \sum_{k=j}^{M} a_{jk} \mathbf{V}_k^n = \sum_{k=1}^{j} b_{jk} \mathbf{W}_k^{n+1}. \tag{19}$$

We can see immediately that when $j = M$, we have

$$\mathbf{V}_M^{n+1} = a_{MM} \mathbf{V}_M^n. \tag{20}$$

Hence,

$$a_{MM} = 1$$

because

$$\| \mathbf{V}_M^{n+1} \| = \| \mathbf{V}_M{}^n \| = 1.$$

We have used the matrix Z, which represents the linear tangential mapping T at the point P_n, in the *canonical basis* $[(e_i); i = 1, M]$.

If we now consider the matrix S, whose columns are the eigenvectors $\mathbf{V}_i{}^n$, then the matrix

$$Z' = S^{-1}ZS$$

represents T^* in the basis $[(\mathbf{V}_i{}^n); i = 1, M]$, i.e., the columns of Z' are the vectors W_i^{n+1}.

The elements of Z' are denoted $(z'_{ij}; i = 1, M; j = 1, M)$. From (19), we derive

$$\mathbf{V}_M^{n+1} = a_{MM}\mathbf{V}_M{}^n = \sum_{k=1}^{M} b_{Mk}\mathbf{W}_k^{n+1}$$

$$= \sum_{l=1}^{M} \left(\sum_{k=1}^{M} b_{Mk}z'_{lk} \right) \mathbf{V}_l{}^n, \tag{21}$$

and also

$$\mathbf{V}_{M-1}^{n+1} = a_{M-1,M}\mathbf{V}_M{}^n + a_{M-1,M-1}\mathbf{V}_{M-1}^n = \sum_{k=1}^{M-1} b_{M-1k}\mathbf{W}_k^{n+1}$$

$$= \sum_{l=1}^{M} \left(\sum_{k=1}^{M-1} b_{M-1k}z'_{lk} \right) \mathbf{V}_l{}^n, \tag{22}$$

$$\mathbf{V}_p^{n+1} = \sum_{l=p}^{M} a_{pl}\mathbf{V}_l{}^n = \sum_{k=1}^{p} b_{pk}\mathbf{W}_k^{n+1}$$

$$= \sum_{l=1}^{M} \left(\sum_{k=1}^{p} b_{pk}z'_{lk} \right) \mathbf{V}_l{}^n, \tag{23}$$

$$\mathbf{V}_1^{n+1} = \sum_{l=1}^{M} a_{1l}\mathbf{V}_l{}^n = {}_{11}\mathbf{W}_1^{n+1}$$

$$= \sum_{l=1}^{M} (b_{11}z'_{l1})\, \mathbf{V}_l{}^n. \tag{24}$$

If we equal the coefficients of the vectors $\mathbf{V}_l^n$ in Eqs. (23), we obtain M relations and $M + 1$ unknown quantities:

$$a_{pp}, \cdots, a_{pM}, b_{p1}, \cdots, b_{pp}$$

We write these quantities as functions of b_{pp} and of the z'_{lj}, and we use the following normalization relation:

$$\| \mathbf{V}_p^{n+1} \| = 1.$$

Since we know all the scalar products $\mathbf{V}_j^n \cdot \mathbf{V}_k^n$, we can compute the unknown quantities, and obtain first, from (18) the new eigenvalue λ_p^{n+1} as a function of the old one:

$$\lambda_p^{n+1} = \lambda_p^n (a_{pp}/b_{pp}), \tag{25}$$

then, from (23) the new eigenvector $\mathbf{V}_p^{n+1}$ in the new basis $[(\mathbf{V}_i^n), i = 1, M]$, as a function of the $(\mathbf{V}_l^n, l = p, M)$. We find that $\mathbf{V}_M^n$ is a vector whose direction is independant of n.
In the particular case $p = M$, we have

$$\lambda_M^{n+1} = \lambda_M^n (a_{MM}/b_{MM}) = \lambda_M^n z'_{MM}$$

We note that we just have to compute the $M/2$ largest eigenvalues, in absolute magnitude. Since the mapping is reciprocal, the others are the inverses of the previous ones.

This new method is an iterative method, which gives the eigenvalues at iteration $n + 1$ as functions of the eigenvalues at iteration n. All the computations are done by means of matrix operations introducing quantities which are all of the same order. And so, the previous numerical difficulties do vanish.

IV. Results

A. *Systems with three degrees of freedom*

M is now equal to 4. In order to start the new method, we have used the old one, described in Section II, in double precision, till the condition

$$| \lambda_1^n | / | \lambda_2^n | \simeq 10^8 \tag{27}$$

is realized; the errors introduced by the two methods are then about the same. This is obtained after a certain number m of iterations. Then, we compute the eigenvectors $(\mathbf{V}_j^m; j = 1, M)$, corresponding to the eigenvalues $(\lambda_j^m, j = 1, M)$,

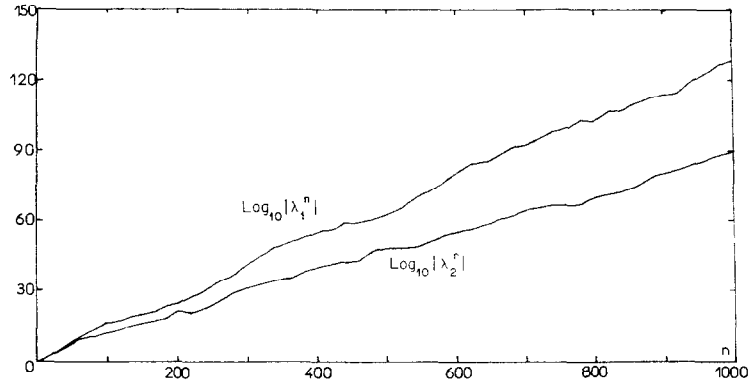

FIG. 3. Upper curve: variation of $\log_{10} | \lambda_1^n |$ with n. Lower curve: variation of $\log_{10} | \lambda_2^n |$ with n. Initial conditions: $x_0 = 2.0$, $y_0 = 0.$, $z_0 = 2.1$, $t_0 = 0$; parameters: $a_1 = -1.3$, $a_2 = -1.3$, $b = 0.01$, $n = 1(1)$ 1000. Computation in double precision, using the new method described in section III.

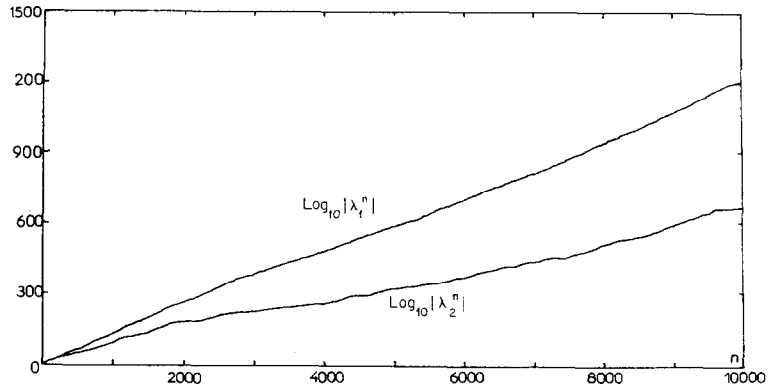

FIG. 4. Same as Fig. 3 but for $n = 1(1)$ 10 000.

and we proceed as described in Section III. Figure 3 shows the results for 1000 iterations. This is to be compared with the results of Figs. 1b, 2a, and 2b. The results on Fig. 4 are obtained for 10 000 iterations. We can see the divergence of the two largest eigenvalues (the logs are plotted) (in absolute magnitude) of the tangential linear mapping of T^n. The two other eigenvalues being the inverses, we then really have two *dilatating* directions and two *contracting* directions, and this is characteristic of a dynamical system similar to a C-system (in the ergodic zone).

We can compare the result on Fig. 4 with the results obtained for the same number of iterations using the previous method, which are plotted on Fig. 5, in

double precision. Then, the observation of two dilatating directions, was impossible. It is due to the precision problems mentioned in Section II. It is to be noted that the new method computes 1000 iterations in about 10 minutes, against one hour in precision 4 using the previous method. We can see on Fig. 6 that there exists an overlapping zone between the two sets of curves plotted in double precision with the two different methods.

B. *Systems with four degrees of freedom*

M is now equal to 6. We did not try to use the multiprecision program because of the prohibitive computer time which would be necessary, but directly applied

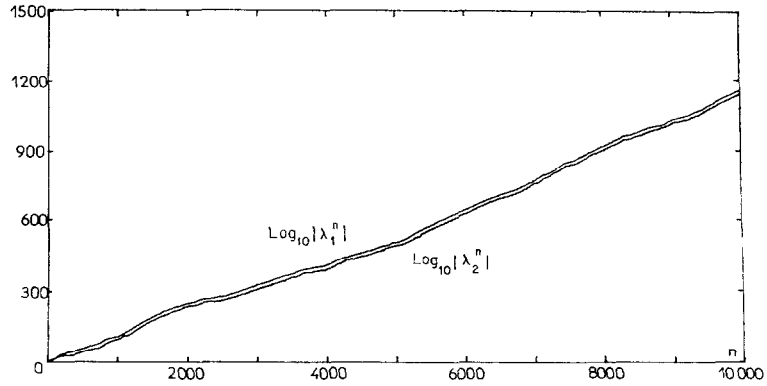

FIG. 5. Same as Fig. 1b, but for $n = 1(1)\ 10\,000$.

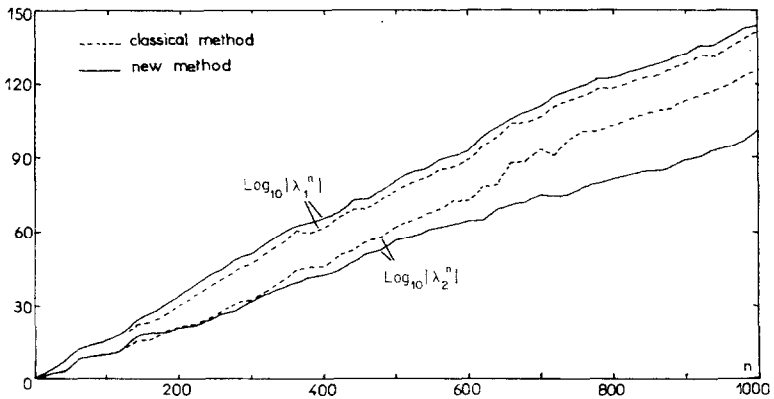

FIG. 6. Same conditions as in Fig. 1b. We show the overlapping zone when using the two different methods; the classical method is used until (27) is satisfied.

the new method to a dynamical system with 4 degrees of freedom. This can be done quite easily, and we use the following mapping:

$$T_6 \begin{cases} x_1 = x_0 + a_1 \sin(x_0 + y_0) + b \sin(x_0 + y_0 + z_0 + t_0 + u_0 + v_0) \\ y_1 = x_0 + y_0 \\ z_1 = z_0 + a_2 \sin(z_0 + t_0) + b \sin(x_0 + y_0 + z_0 + t_0 + u_0 + v_0) \\ t_1 = z_0 + t_0 \\ u_1 = u_0 + a_3 \sin(u_0 + v_0) + b \sin(x_0 + y_0 + z_0 + t_0 + u_0 + v_0) \\ v_1 = u_0 + v_0 \end{cases} \pmod{2\pi} \quad (28)$$

which is a generalisation of the mapping T.

We could have shown, as we did at the beginning of Section II, that the characteristic equation of the tangential linear mapping T_6^{n*}, is *reciprocal.* Now, we study the behavior of the three largest eigenvalues (in absolute magnitude) of the tangential linear mapping T_6^{n*}.

We use in double precision the classical method until we have

$$| \lambda_1^n | / | \lambda_2^n | \simeq 10^8 \qquad \text{and} \qquad | \lambda_2^n | / | \lambda_3^n | \simeq 10^8,$$

and then we proceed as stated in Section IV A.

Figure 7 shows that we have three *dilatating* directions, and therefore three

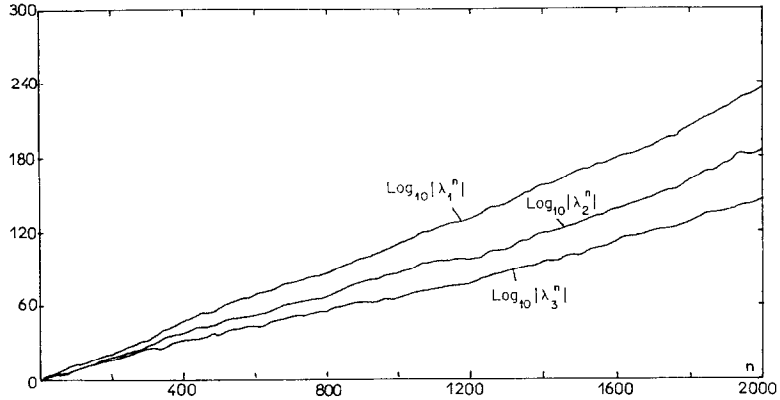

FIG. 7. Eigenvalues of the six-dimensional mapping (28). Variations of $\log_{10} | \lambda_1^n |$, $\log_{10} | \lambda_2^n |$, $\log_{10} | \lambda_3^n |$, with n. Initial conditions: $x_0 = 2.0$, $y_0 = 0.$, $z_0 = 2.1$, $t_0 = 0.$, $u_0 = 2.0$, $v_0 = 0.$; parameters: $a_1 = -1.3$, $a_2 = -1.3$, $a_3 = -1.3$, $b = 0.05$, $n = 1(1)\ 2000$. Computation using the new method.

contracting directions, which are characteristic, as seen before, of a dynamical system near of a C-system, in the ergodic zone. This method will be a tool for the study of the stochasticity of such systems. It can be easily extended to $N = 5, 6, \ldots$. The only limitation is computational time.

V. Conclusion

The methods described in this paper are of two kinds: multiprecision and special numerical treatment. The first method gives only some indications concerning the divergence of the second eigenvalue of the linear tangential mapping T^{n*} of T^n. The second method shows numerically that the dynamical system, equivalent to the multidimensional mapping T, is near of a C-system in the ergodic zone. The fact that this method can be generalized without difficulties to systems with a small number N of degrees of freedom ($N = 4, 5, 6$) is important. This method can be used as a numerical tool for the study of the stochasticity of such dynamical systems. More generally, this method can be used for any physical problem introducing the computation of eigenvalues whose ratios (in absolute magnitude and decreasing order) are greater than the inverse of the relative error of the computer.

VI. Acknowledgment

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